Complex Analysis Study Guide Test 1 (Draft!!)

The test is Friday February 17th. No calculators. Closed notes.

Material for test 1:

- Complex numbers: know what the complex plane is, how to describe complex numbers, and how to do arithmetic on them.
- Be familiar with the three forms of representing complex numbers, and be able to switch between them when necessary. (Of course it is assumed you know the unit circle)
- Be able to find roots and powers of complex numbers. Understand why roots are multidefined and how to find all the values.
    - Understand how this relates to roots of unity. (Of course you must know what roots of unity are)
- Be familiar with how to extend notions on $\mathbb{R}$ to $\mathbb{C}$, such as:
    - Limits -> Complex limits
    - Derivatives -> Complex derivatives
    - Differentiable function -> Analytic function
    - Intervals -> Regions
    - Absolute value -> Complex absolute value
    - Continuity -> Complex continuity
    - Functions -> Multivalued functions
- Be familiar with important complex functions:
    - The exponential function e^z
    - Complex trig functions: $\sin(z)$, $\cos(z)$
    - Hyperbolic trig functions: $\sinh(z)$, $\cosh(z)$
    - Polynomials
    - Rational functions
    - The complex logarithm
- Understand the idea of stereographic projection and compactification

Practice problems

Below are a selection of problems from our textbooks that looks like reasonable problems that could appear on a test. An “easy” problem means that you should be able to jump right in and start solving it immediately. A “medium” problem means it is expected that you’ll need to think a little before solving the problem. A “hard” problem means you’ll need to think a lot and maybe work out some details before solving the problem.

Easy Problems

1. Let $z_1 = 2 + i, z_2 = 3 - 2i$. Find $|3z_1 - 4z_2|$. (1.2.a)
2. Express $-\sqrt{6} - i\sqrt{2}$ in both trigonometric form and exponential form. (1.16.c)
3. Find $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{10}$. (1.26.c)
4. Solve $z^5 = -32$ (1.28)
5. Let $z = x + iy$ where $x, y \in \mathbb{R}$. Show that $|e^z| = e^x$. (2.8b)
6. Divide $3z^4 - 2z^3 + 8z^2 - 2z + 5$ by $z - i$. (Might be too long for a test; haven't worked it out myself)
7. Find $\lim_{z \to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$. (2.25)
8. Find $\lim_{z \to -2i} \frac{(2z+3)(z-1)}{z^2 - 2z + 4}$. (2.29b)
9. Find $\frac{d}{dz}(z^3 - 2z)$ (3.1)
10. Find $\frac{d}{dz}\cos^2(2z + 3i)$ (3.17)
11. Determine where $f(z) = \frac{z}{z^2+4}$ is singular. (3.25)
12. Show that $f(z) = \sin(2z)$ is analytic on the entire complex plane. (3.46)
13. Show that $\mathrm{Re}(z) \leq |z|$ (R 1.4.d)
14. Sketch the region given by $|2z - 4| \leq 2$ (R1.2.2.d)
15. An analytic function $f(z) = u(x, y) + iv(x, y)$ has $u(x, y) = 3x^2y - y^3$. Find $f(z)$ as precisely as you can. (R2.1.2.a)

Medium problems

1. Graph $\left|\frac{z-3}{z+3}\right| = 2$. (1.48)
2. Solve $z^2(1 - z^2) = 16$ (1.50)
3. Show that the function $f(z) = z^{\frac{1}{5}}$ is multivalued. (2.6)
4. Derive the derivative of $f(z) = \sin(z)$. (3.12a)
5. Given the power series expansion for $\cosh(z)$, find the power series expansion for $\frac{\cosh(z)-1}{z^2}$ (R1.2.5)
6. What does the unit square in the first quadrant map to under $f(z) = \frac{1}{z}$? (R1.2.7)
7. Show that $f(z) = \mathrm{Im}(z)$ is not analytic. (R1.3.5)
8. Use the series expansion of e^x to show that $\lim_{z \to 0} e^z - (1 + z) = 0$. (R1.3.9)
9. Find all branch points for the multidefined function $f(z) = \frac{1}{(z-1)^{\frac{1}{2}}}$ (R2.2.1.a)
10. Find a branch cut for the multidefined function $f(z) = 1/(z - 1)^{\frac{1}{2}}$ (R2.2.1.a)
11. Solve $3 + 2e^{z-i} = 1$ for z. (R2.2.3.b)

Hard problems

1. Find all branch points for the multidefined function $f(z) = \log\big((z - 1)(z - 2)\big)$. (R2.3.2.a)
2. Find a branch cut for the multidefined function $f(z) = \log\big((z - 1)(z - 2)\big)$. (R2.3.2.a)